A Note on the Construction of Ordinal Numbers

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In the present, we have some systems of ordinal numbers which are constructed through some finite procedures. For example, (1) the system of ordinal numbers less than ε_0, which are constructed by 1 and operations $\alpha+\beta$ and ω^α, (2) so-called the system of Ackerman's ordinal numbers, which are constructed by 1 and operations $\alpha+\beta$ and, (α, β, γ), (3) the system of ordinal diagrams, which are constructed by elements of given well ordered sets, and operations $\alpha \# \beta$ (natural sum) and (α, β, γ) or (α, β).

In these constructions, the operation sum $\alpha+\beta$ or natural sum $\alpha \# \beta$ have a special role, and in some cases, if it were not contained in the construction, to prove that the system is well ordered would be performed mere easily. In this stand point, we give a following note on the construction of ordinal numbers.

A set S of primitive symbols

$$a, b, \cdots \in S,$$

a set P of primitive operations

$$p(*, \cdots, *), \qquad q(*, \cdots, *), \cdots \in P,$$

a symbol for natural sum $\#$ are treated.

System A:

(1) An element of S is an element of A.

(2) If an n-ary operation $p(*, \cdots, *)$ is an element of P, and $\alpha_1, \cdots, \alpha_n$ are elements of A, then a figure

$$p(\alpha_1, \cdots, \alpha_n)$$

is an element of A.

(3) If α_1 and α_2 are elements of A, then a figure

$$\alpha_1 \# \alpha_2$$

is an element of A.

If the last step of the construction of an element ρ of A is (1) or (2), ρ is called to be connected.

System B:

(1′) An element of S is an element of B.

(2′) If an n-ary operation $p(*, \cdots, *)$ is an element of P, and $\alpha_1, \cdots, \alpha_n$ are elements of B, then a figure

$$p(\alpha_1, \cdots, \alpha_n)$$

is an element of B.

Of course, B is a proper subset of A.

We suppose that $<$ is a linear-ordering for the system A, and is a well-ordering for the system which is the image by the canonical mapping of B into A. Moreover we suppose the following conditions on $<$ in A:

Let $\varphi(\alpha)$ be an operation (does not need that φ is a primitive operation, but a combination of operations or the identity) for α.

(a) $\varphi(\alpha \# \beta) = \varphi(\beta \# \alpha)$,

(b) $\varphi(\alpha) < \varphi(\alpha \# \beta)$,

(c) if ρ is connected and $\varphi(\alpha) < \varphi(\rho)$, and $\varphi(\beta) < \varphi(\rho)$, then $\varphi(\alpha \# \beta) < \varphi(\rho)$.

(d) when $\varphi(\alpha) = \psi(\sigma)$, $\varphi(\alpha \# \alpha) < \psi(\sigma \# \sigma)$, and σ is connected, then $\varphi(\alpha \# \alpha \# \cdots \# \alpha) < \psi(\sigma \# \sigma)$.

Then, $<$ is a well-ordering for A.

Proof:

When $\varphi(\alpha)$ is any operation, $\varphi(\alpha \# \beta)$ is called a convolution of $\varphi(\alpha)$ and $\varphi(\beta)$. We denote M^c, for a subset M of A, the convolution-closure of M. If B_ρ is the section of B by ρ which is an element of B, then

$$A_\rho = B_\rho^c = \{\alpha \mid \alpha < \rho\} .$$

For any element α of A, we shall define the skeleton $\tilde{\alpha}$ of α which is an element of B.

If α is an element of B, then $\tilde{\alpha}$ is α itself.

If α is of the form

$$\varphi(\rho_1 \# \rho_2 \# \cdots \# \rho_i \# \cdots \# \rho_n) ,$$

and for every $\rho_j (1 \leqq j \leqq n)$

$$\varphi(\rho_j) \leqq \varphi(\rho_i)$$

then $\tilde{\alpha}$ is $\tilde{\varphi}(\rho_i)$, where $\rho_1, \cdots, \rho_n$ are connected. Then for any element α of A, there exists an element β of $\{\tilde{\alpha}\}^c$, such that $\alpha < \beta$.

We shall prove that A_ρ is well ordered by the transfinite induction for ρ in B. It is sufficient to prove that if ρ in B is accessible in A, then every element of $\{\rho\}^c$ is accessible in A.

Let $\pi_1, \pi_2, \cdots, \pi_n$ be all convolution of ρ and ρ, and

$$\pi_1 < \pi_2 < \cdots < \pi_n ,$$

where π_i is of the form $\varphi_i(\tau_i \# \tau_i)$, and $\varphi_i(\tau_i) = \rho$. If $\gamma < \pi_1$, then there exists δ such that $\gamma < \varphi_1(\tau_1 \# \delta)$ and $\varphi_1(\delta) \leqq \varphi_1(\tau_1)$. By the transfinite induction until $\varphi_1(\delta)$, $\varphi_1(\tau_1 + \delta)$ is accessible. By the induction $\varphi_1(\tau_1 \# \cdots \# \tau_1)$

is accessible. Similarly, $\pi_2, \cdots, \pi_n$ are accessible.

Therefore, any element of $\{\rho\}^c$ is accessible.

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